Cláudio R. Ávila da Silva, Jr.

Giuseppe Pintaude
e-mail: giuseppepintaude@gmail.com

Department of Mechanics (DAMEC),
Federal University of Technology-Paraná (UTFPR),
Avenida Sete de Setembro,
3165, Curitiba,
Paraná 80230-901, Brazil

Hazim Ali Al-Qureshi

Marcelo Alves Krajnc

Computational Applied Mechanics Group (GMAC),
Federal University of Santa Catarina (UFSC),
P.O. Box 476, Florianópolis,
Santa Catarina 88040-900, Brazil

An Application of Mean Square Calculus to Sliding Wear

In this paper the Archard model and classical results of mean square calculus are used to derive two Cauchy problems in terms of the expected value and covariance of the worn height stochastic process. The uncertainty is present in the wear and roughness coefficients. In order to model the uncertainty, random variables or stochastic processes are used. In the latter case, the expected value and covariance of the worn height stochastic process are obtained for three combinations of correlation models for the wear and roughness coefficients. Numerical examples for both models are solved. For the model based on a random variable, a larger dispersion in terms of worn height stochastic process was observed. [DOI: 10.1115/1.3173603]

Keywords: constitutive modeling of materials, mechanical properties of materials

1 Introduction

In his article about the history of the Associated Electrical Industries research laboratory, Hirst [1] referred to the Archard wear model as "the total amount of wear, then, depended on the probability that a wear particle would be produced at an encounter and, if it did so, on the size of that particle." Although a probabilistic approach can be used for the wear coefficient, it has in fact usually been treated deterministically, and a number of practical applications using deterministic approaches can be found in the literature [2–4]. Indeed, descriptions of the use of probabilistic approaches to determine wear coefficient are rarely found in the literature [5,6]. Usually, models that predict wear rate make use of this coefficient together with failure concepts to define the predominant wear mechanism [7]. Basic concepts of low cycle fatigue [8], fracture mechanics [9], and dislocation theory [10] have been used for this purpose.

Ávila da Silva and Pintaude [11] described a probabilistic version of the Archard equation and modeled the uncertainty in the wear coefficient. They obtained approximate solutions for the worn height stochastic process and used the Karhunen–Loève series to represent the uncertainty in the wear coefficient. They also considered the contact area in the initial value problem (IVP), allowing a relationship to be established between this concept and the wear rate. According to Kapoor et al. [12], this kind of relationship is fundamental when modeling mild wear in sliding systems. Although Ávila da Silva and Pintaude [11] did not consider the contact area as a stochastic process, the probabilistic approach is extensively applied in mechanical contact models that incorporate surface roughness [13,14].

This paper contributes to the model proposed by Archard, by describing the use of an approach based on differential equations, whose exact solutions are the first and second statistical moments of the worn height stochastic process. The formulation of the wear problem is based on the stochastic version of the Archard model described by Ávila da Silva and Pintaude. The uncertainty is assumed to be in the roughness and wear coefficients, and these are modeled by means of random variable or stochastic processes.

2 The Archard Model as an IVP

Ávila da Silva and Pintaude [11] described the following IVP based on the Archard model for the worn height h

$$\frac{dh}{dt}(t,\omega) = \upsilon(t,\omega), \qquad \forall\ (t,\omega) \in (0,T) \times \Omega$$
$$h(0) = h_0 \tag{1}$$

where $\upsilon(t,\omega)=k(t,\omega)\cdot\beta(t,\omega)\cdot v(t,\omega)$, wherein k, β, and v are the wear coefficient, roughness coefficient, and sliding velocity, respectively. The solution to the IVP is the worn height stochastic process (WHSP), whose domain is defined in $(0,T)\times\Omega$, with $(0,T)\subset\mathbb{R}$ as a restricted interval; ω is a generic element of the sampling space of events Ω. In the case of the wear coefficient, ω can be physically explained as a random event associated to the asperities density on the sliding surfaces. In this way, the sampling space of events Ω would be the set of all possible random events related to the asperities density. The roughness coefficient is defined as

$$\beta = \frac{A_y}{A_n} \tag{2}$$

where A_y and A_n are the plastic and apparent contact areas, respectively. As the Archard model predicts that plastic deformation must take place for the material to be removed, a portion of the contact area is assumed to be under full plastic deformation. The solution to the IVP defined in Eq. (1) is of the right hand side (RHS) type, following the classification given by Soong [15] for stochastic differential equations. From Eq. (1) and the classical results of probability theory, the Cauchy problems for the expected value and covariance will be formulated. The following hypotheses are required to ensure these problems are well-defined in the Hadamard sense:

(H1) k, β, and v are continuous in mean square.
(H2) k, β, and v are uncorrelated.

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(H3) $\exists \underline{k}, \bar{k}, \underline{\beta}, \bar{\beta}, \underline{v}, \bar{v} \in \mathbb{R}^{+}$, so that $P(\{(t,\omega) \in [0,T] \times \Omega \mid k(t,\omega) \in [\underline{k}, \bar{k}] \wedge \beta(t,\omega) \in [\underline{\beta}, \bar{\beta}] \wedge v(t,\omega) \in [\underline{v}, \bar{v}]\}) = 1$.

Hypothesis H1 provides that dh/dt be continuous in mean square, making it feasible to use theoretical results regarding mean square continuity. Hypothesis H2 becomes reasonable if one assumes that no transition between any wear mechanisms occurs for the problems being considered. From an experimental point of view, the results given by Welsh [16] indicate that there is an extensive range of sliding velocities (0.017–2.66 m/s) for which no change in the values of k and β takes place. Thus, the deviations in k, β, and v do not result in mechanisms that would in turn lead to a statistical correlation between these three parameters, hence justifying H2. Finally, H3 ensures that the probability distributions for the sliding velocity and wear and roughness coefficients can only be uniform. This hypothesis ensures that the source term of Eq. (1) is contained within $[0,T] \times \Omega$.

3 Notations, Definitions, and Corollaries

In this section some basic definitions and results of stochastic processes used to develop the formulations described in this paper are presented. The definitions and demonstrations can be found in Refs. [17,15]. For the purposes of the following definitions, the probability space $(\Omega, \mathcal{F}, P)$ must be established. This is a measurement space defined in a σ-algebra $\mathcal{F}$ from the sample space Ω and the measure P, called probability measure, or probability. The solution to Eq. (1) is the WHSP $h:[0,T] \times \Omega \to \mathbb{R}$, which is a function in two variables $(t,\omega) \in [0,T] \times \Omega$ belonging to real values and P-measurable as a function of $\omega \in \Omega$ for each fixed $t \in [0,T]$.

DEFINITION (*expected value*). *The expected value* μ_h of the stochastic process $h:[0,T] \times \Omega \to \mathbb{R}$ is given by

$$\mu_h(t) = \langle h(t,\omega) \rangle = \int_{\Omega} h(t,\omega) dP(\omega) \quad (3)$$

where $dP(\omega)$ is the probability measurement and $\langle \cdot \rangle$ is the mathematical expectation operator.

DEFINITION (*autocorrelation*). *The autocorrelation* R_h of the stochastic process $h:[0,T] \times \Omega \to \mathbb{R}$ is defined by

$$R_h(s,t) = \langle h(s,\omega),(t,\omega') \rangle = \int \int_{\Omega \times \Omega} h(s,\omega) h(t,\omega') dP(\omega,\omega') \quad (4)$$

DEFINITION (*continuity in mean*). *A stochastic process* $h:[0,T] \times \Omega \to \mathbb{R}$ of second order, i.e., $\langle h^2(t,\omega) \rangle < \infty \, \forall \, t \in [0,T]$, is continuous in mean or even mean square continuous for a fixed $t \in [0,T]$ if the following limit exists:

$$\lim_{\tau \to 0} h(t+\tau,\omega) = h(t) \quad \text{or even}$$

$$\lim_{\tau \to 0} \langle [h(t+\tau,\omega) - (t,\omega)]^2 \rangle = \lim_{\tau \to 0} \int_{\Omega} [h(t+\tau,\omega) - h(t,\omega)]^2 dP(\omega) = 0 \quad (5)$$

DEFINITION (*derivative in mean*). *A stochastic process* $h:[0,T] \times \Omega \to \mathbb{R}$ of second order has a derivative in mean or mean square derivative at $t \in [0,T]$ if the following limit exists:

$$\frac{dh}{dt}(t+\tau,\omega) = \lim_{\tau \to 0} \frac{h(t+\tau,\omega) - h(t+\tau,\omega)}{\tau} \quad (6)$$

COROLLARY. *If* $dh/dt(t,\omega)$ exists, then $R_{dh/dt}(s,t) = \partial^2 R_h / \partial s \, \partial t(s,t)$ in $[0,T] \times [0,T]$.

Proof. The demonstration of this corollary can be seen in Ref. [17].

The result shown in this corollary is the basis upon which the formulations in this paper are developed.

4 Formulation of the Cauchy Problem for the Expected Value and Covariance

In this section, the formulations in terms of the Cauchy problems for the expected value and covariance of the WHSP for a sliding system based on the Archard model are described. Fixing $t \in (0,T)$ in Eq. (1), one can obtain the random variable $dh/dt(t,\omega)$, and applying mathematical expectation yields

$$\left\langle \frac{dh}{dt}(t,\omega) \right\rangle = \langle v(t,\omega) \rangle \quad (7)$$

From H1 and its corollary one can obtain

$$\left\langle \frac{dh}{dt}(t,\omega) \right\rangle = \frac{d\mu_h}{dt}(t) \quad (8)$$

and

$$\langle v(t,\omega) \rangle = \mu_v(t) \quad (9)$$

where μ_v is the expected value of the random variable v. From Eqs. (7)–(9) it follows that

$$\frac{d\mu_h}{dt} = \mu_v \quad (10)$$

From Eq. (10) and the initial condition given in Eq. (1), the following IVP can be proposed for the expected value of the worn height

$$\frac{d\mu_h}{dt} = \mu_v, \quad \forall \, t \in (0,T)$$
$$\mu_h(0) = \mu_{h(0)} \quad (11)$$

The solution to the IVP in Eq. (11) is obtained by integrating and applying the initial condition

$$\mu_h(t) = \mu_{h(0)} + \int_0^t \mu_v(s) ds, \quad \forall \, t \in [0,T] \quad (12)$$

It should be mentioned that in order to get the expected value of the worn height, one should know, if possible, the correlation between the wear and roughness coefficients and the sliding velocity.

The autocorrelation of the WHSP is obtained in a similar fashion. Fixing $s,t \in (0,T)$ in Eq. (1), the random variables $dh/dt(s,\omega)$ and $dh/dt(t,\omega)$ are generated. Taking the mathematical expectation of the product of these variables yields

$$\left\langle \frac{dh}{dt}(s,\omega), \frac{dh}{dt}(t,\omega) \right\rangle = \langle v(s,\omega), v(t,\omega) \rangle \quad (13)$$

The left hand side of Eq. (13) corresponds to the autocorrelation function of random variables $dh/dt(s,\omega)$ and $dh/dt(t,\omega)$

$$R_{dh/dt}(s,t) = \left\langle \frac{dh}{dt}(s,\omega), \frac{dh}{dt}(t,\omega) \right\rangle \quad (14)$$

where $R_{dh/dt}$ is the autocorrelation function of the random variables $dh/dt(s,\omega)$ and $dh/dt(t,\omega)$. From Eq. (1) and hypothesis H1, it is known that that v is continuous in mean square, implying that dh/dt is also continuous in mean square. From the corollary it follows that

$$R_{dh/dt}(s,t) = \frac{\partial^2 R_h}{\partial s \, \partial t} \quad (15)$$

Putting Eq. (15) in Eq. (13) gives

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$$\frac{\partial^2 R_h}{\partial s\,\partial t}(s,t) = \langle \upsilon(s,\omega), \upsilon(t,\omega)\rangle \tag{16}$$

From H2 it follows that

$$\langle \upsilon(s,\omega), \upsilon(t,\omega)\rangle = \langle k(s,\omega),k(t,\omega)\rangle\langle \beta(s,\omega),\beta(t,\omega)\rangle\langle v(s,\omega),v(t,\omega)\rangle \tag{17}$$

which may be expressed as

$$\langle k(s,\omega),k(t,\omega)\rangle\langle \beta(s,\omega),\beta(t,\omega)\rangle\langle v(s,\omega),v(t,\omega)\rangle = (R_k \cdot R_\beta \cdot R_v)(s,t) \tag{18}$$

From Eqs. (16)–(18) one can obtain

$$\frac{\partial^2 R_h}{\partial s\,\partial t}(s,t) = (R_k \cdot R_\beta \cdot R_v)(s,t), \quad \forall\ (s,t) \in (0,T) \times (0,T) \tag{19}$$

The autocorrelation function of WHSP is determined by integrating Eq. (19). The initial conditions associated with Eq. (19) are obtained based on the initial condition of the IVP defined in Eq. (1). If $t \in [0,T]$, the autocorrelation of the random variables $h(0)$ and $h(t)$ is given by

$$R_h(0,t) = \langle h(0),h(t)\rangle = h_0\langle h(t)\rangle = h_0 \cdot \mu_h(t) \tag{20}$$

and from the symmetry of the autocorrelation function, one can derive

$$R_h(s,0) = h_0 \cdot \mu_h(s) \tag{21}$$

Equations (19)–(21) define the problem for the autocorrelation function of WHSP

$$\begin{gathered}\frac{\partial^2 R_h}{\partial s\,\partial t} = (R_k \cdot R_\beta \cdot R_v)(s,t), \quad \forall\ (s,t) \in (0,T) \times (0,T) \\ R_h(s,0) = h_0 \cdot \mu_h(s) \\ R_h(0,t) = h_0 \cdot \mu_h(t), \quad \forall\ s,t \in [0,T]\end{gathered} \tag{22}$$

The solution to Eq. (22) is derived directly by integrating and applying the initial conditions. For the case where $h_0 \neq 0$, the problem defined in Eq. (11) must be solved. It is important to observe that the problems defined in Eqs. (11) and (22) are linear in terms of the expected value and autocorrelation of the WHSP, respectively. In the Sec. 5, the formal solutions for the problems defined in Eqs. (11) and (22) will be described for the cases where the uncertainty is modeled by means of a random variable or stochastic process.

5 Uncertainty Modeling

In the Archard model, the uncertainty is considered to be in the wear and roughness coefficients. Two different approaches to modeling this uncertainty will be analyzed: One using a random variable and another using a stochastic process. In the former approach, the variable $\zeta:\Omega \to \mathbb{R}$ is defined as

$$\zeta(\omega) = \mu_\zeta + \xi(\omega) \tag{23}$$

where μ_ζ is the expected value of the random variable $\zeta(\omega)$, which has a probability density distribution F_ξ. The random variable $\xi(\omega)$ has the following properties:

$$\langle \xi\rangle = 0 \wedge \langle \xi^2\rangle = \sigma_\zeta^2 \tag{24}$$

where σ_ζ^2 is the variance of the random variable $\zeta(\omega)$.

A weakly stationary stochastic process defined as follows will be used for the case where the uncertainties in the wear and roughness coefficients are modeled by stochastic processes

$$\zeta(t,\omega) = \mu_\zeta \cdot [1 + g_\zeta(t,\omega)] \tag{25}$$

where μ_ζ is the expected value, and g_ζ a stochastic process with the following properties

$$\langle g_\zeta(t)\rangle = 0, \quad \forall\ t \in [0,T] \wedge \langle g_\zeta(s),g_\zeta(t)\rangle = \frac{1}{\mu_\zeta^2} \cdot C_\zeta(s - t), \quad \forall\ (s,t) \in [0,T] \times [0,T] \tag{26}$$

so that C_ζ is the covariance function of the stochastic process ζ.

The choice of the model described in Eq. (25) is based on experimental observation of the run-in process. According to Blau [18], most tribological transitions can be classified as nonperiodic processes, allowing the steady-state regime to be predicted. In the present study, modeling the run-in period is not a goal, so that, as in Eq. (25), the expected value (μ_ζ) becomes independent of time, since the steady-state wear regime has been reached. These considerations apart, the model is suitable for use with other kinds of variations over time.

6 Expected Value and Covariance of the WHSP

In this section the formal solutions of the problems defined in Eqs. (11) and (22) are given for the uncertainty models referred to in the Sec. 5. These solutions are determined by integrating and applying the initial conditions. As mentioned in Eq. (1), all the terms on the right hand side can be modeled as stochastic processes. This approach would be useful for systems in which wear is caused by erosion, where the velocity of the particles is not only an important factor in wear rates but also difficult to measure [19]. In contrast, in sliding systems, the velocity usually varies very little and is relatively easy to control [20]. Only sliding systems will be considered here. Thus, the sliding velocity is assumed to be constant and deterministic, i.e., $P(\{(t,\omega) \in [0,T] \times \Omega \mid v(t,\omega) = v\}) = 1$. In addition, the initial condition is assumed to be homogeneous and deterministic, i.e., $P(\{\omega \in \Omega \mid h(0,\omega) = 0\}) = 1$. To this end, the initial height of the worn specimen should be normalized to give a reduction in height of zero ($h_0 = 0$). Some examples of similar normalization procedures can be found in Ref. [21].

6.1 Modeling Using a Random Variable.

From the description of the uncertainty defined in Eq. (23) for the wear and roughness coefficients in Eq. (11), it follows that

$$\begin{gathered}\frac{d\mu_h}{dt}(t) = \mu_k \cdot \mu_\beta \cdot v, \quad \forall\ t \in (0,T) \\ \mu_h(0) = 0\end{gathered} \tag{27}$$

Integrating and applying the initial condition yields the expected value function of the WHSP

$$\mu_h(t) = \mu_k \cdot \mu_\beta \cdot v \cdot t, \quad \forall\ t \in [0,T] \tag{28}$$

If the expected value and autocorrelation functions are known, the covariance of the worn height can be obtained. Thus, the solution to the problem defined in Eq. (22) is required. The autocorrelation for the case where the uncertainty in the wear and roughness coefficients is modeled by random variables is obtained by imposing the uncertainty model given in Eq. (23) and its properties (Eq. (24)) on Eq. (22)

$$\begin{gathered}\frac{\partial^2 R_h}{\partial s\,\partial t}(s,t) = (\mu_k^2 + \sigma_k^2) \cdot (\mu_\beta^2 + \sigma_\beta^2) \cdot v^2, \quad \forall\ (s,t) \in (0,T) \times (0,T) \\ R_h(s,0) = 0 \\ R_h(0,t) = 0, \quad \forall\ s,t \in [0,T]\end{gathered} \tag{29}$$

The solution to the problem shown in Eq. (29) is given by

$$R_h(s,t) = (\mu_k^2 + \sigma_k^2) \cdot (\mu_\beta^2 + \sigma_\beta^2) v^2 st, \quad \forall\ (s,t) \in [0,T] \times [0,T] \tag{30}$$

Subtracting the term $(\mu_k \cdot \mu_\beta \cdot v)^2 \cdot s \cdot t$ from Eq. (30) gives the covariance function of the WHSP

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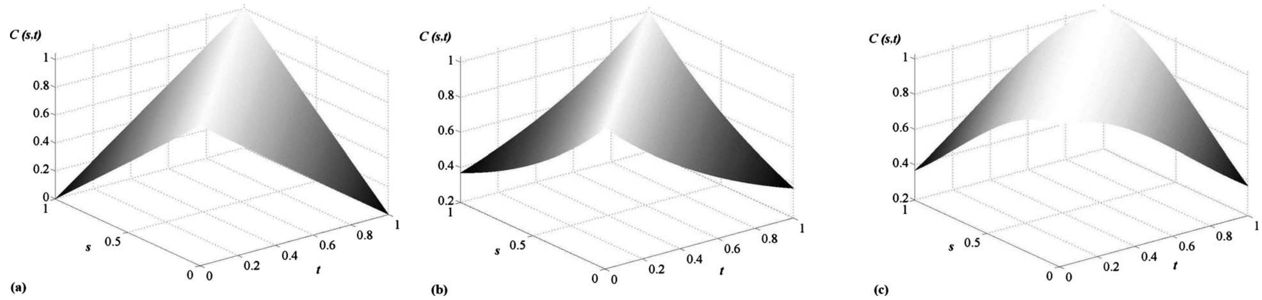

Fig. 1 Graphs of types (a), (b), and (c) covariance functions defined in $[0,1]\times[0,1]$

$$C_h(s,t) = (\mu_k^2\sigma_\beta^2 + \mu_\beta^2\sigma_k^2 + \sigma_k^2\sigma_\beta^2)\cdot v^2 \cdot s \cdot t, \quad \forall\,(s,t) \in [0,T]\times[0,T] \tag{31}$$

Analysis of Eq. (31) reveals that the covariance of the WHSP is explicitly defined in terms of the expected values and standard deviations of the wear and roughness coefficients.

6.2 Modeling Using a Stochastic Process. In this section the expected value and covariance functions for the WHSP are described for the case defined in Eq. (25), where the uncertainties in the wear and roughness coefficients are modeled as stochastic processes. Using the properties of these stochastic processes (Eq. (26)), the expected value of the WHSP is then given by Eq. (28). The autocorrelation function is determined by solving the following problem

$$\begin{aligned} &\frac{\partial^2 R_h}{\partial s\,\partial t}(s,t) = (\mu_k\cdot\mu_\beta\cdot v)^2[1 + (C_k\cdot C_\beta)(s-t)],\\ &\forall\,(s,t) \in (0,T)\times(0,T)\\ &R_h(s,0) = 0\\ &R_h(0,t) = 0, \quad \forall\, s,t \in [0,T] \end{aligned} \tag{32}$$

where C_k and C_β are the covariance functions of the stochastic processes for the wear and roughness coefficients, respectively. The solution to Eq. (32) is obtained by integrating and applying the initial conditions

$$R_h(s,t) = v^2\int_0^t\int_0^s [(\mu_k\cdot\mu_\beta)^2 + (C_k\cdot C_\beta)(u-w)]dudw, \quad \forall\,(s,t) \in [0,T]\times[0,T] \tag{33}$$

The covariance function of the WHSP is

$$C_h(s,t) = v^2\int_0^t\int_0^s (C_k\cdot C_\beta)(u-w)dudw, \quad \forall\,(s,t) \in [0,T]\times[0,T] \tag{34}$$

The variance function is obtained from Eq. (34) by setting the integration limits to $s=t$, yielding $V_h(t)=C_h(t,t)$. It should be noted that both the problems defined in Eqs. (29) and (32) are linear in terms of the autocorrelation function. Sec. 7 gives numerical examples where the stochastic processes have the following types of covariance:

(a) $C(s,t)=1-|s-t|/\delta, \quad \forall\,(s,t)\in[0,T]\times[0,T]$
(b) $C(s,t)=(1+|s-t|/\delta)\cdot e^{-\alpha|s-t|}, \quad \forall\,(s,t)\in[0,T]\times[0,T]$
(c) $C(s,t)=e^{-\alpha(s-t)^2}, \quad \forall\,(s,t)\in[0,T]\times[0,T]$

Figure 1 shows the graphs of the covariance functions for the three types of covariance (a), (b), and (c). The functions are defined in $[0,1]\times[0,1]$, with a correlation parameter $\alpha=\delta=1$.

Figure 2 shows the graphs of types (a), (b), and (c) covariance functions defined in $[-1/\sqrt{2},1/\sqrt{2}]$ with $\tau=s-t$. It can be seen

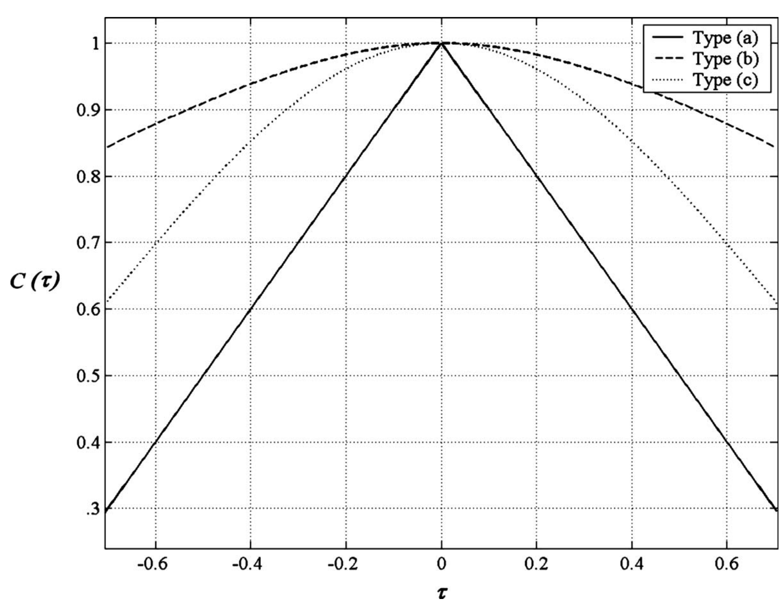

Fig. 2 Graphs of types (a), (b), and (c) covariance functions defined in $[-1/\sqrt{2},1/\sqrt{2}]$

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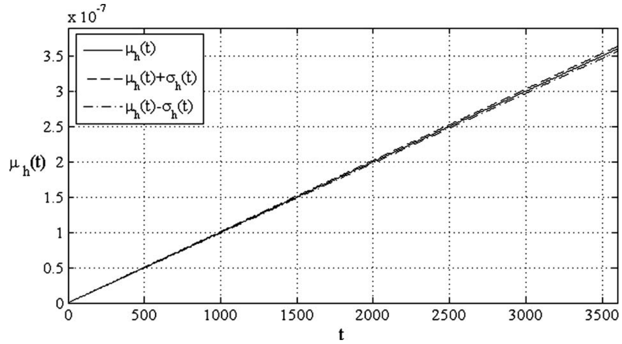

Fig. 3 Expected value of the worn height stochastic process

that the type (a) covariance function has the smallest support, supp(C), and that the type (b) function has the smallest correlation decay. Note that a stochastic process with this kind of correlation gives rise to strongly related random variables.

In both cases of uncertainty modeling, the variance of the WHSP is determined by making $V_h(t)=C_h(t,t)$ with $t\in[0,T]$ in Eqs. (11) and (22).

7 Numerical Results

In this section two sliding wear problems with uncertainty in the wear and roughness coefficients are described. The methodologies discussed in Secs. 4–6 are used, and the uncertainties in the wear and roughness coefficients are modeled by random variables or stochastic processes. The expected value and covariance functions of the WHSP are obtained as solutions of the problems defined in Eqs. (28), (31), and (34). To compare the results of the two different modeling approaches, the same values for the expected value and standard deviation are used. These were previously described in Ref. [11].

- Expected value and standard deviation of wear coefficient $\mu_k=10^{-7}$ and $\sigma_k=10^{-8}$
- Expected value and standard deviation of roughness coefficient $\mu_\beta=10^{-5}$ and $\sigma_\beta=10^{-6}$
- Sliding velocity $v=1$ m/s
- Sliding time $T=3600$ s

Problem 1: Wear and roughness coefficients modeled by random variables. In this problem the wear and roughness coefficients are modeled as uniform random variables. Figure 3 shows the graph of the expected value function of the WHSP. It should be noted that because of the definitions and properties given in Eqs. (23)–(26), the expected value of the WHSP is the same for both the random variable and stochastic process models.

In Fig. 3 one can observe a small variability of the worn height stochastic process in relation to the curve that represents the expected value. In this example, for short intervals of sliding time, the deterministic modeling of wear problem implies in a good approximation for the stochastic modeling of the problem. On the other hand, for larger intervals of time, the dispersion increases, as mentioned in Ref. [22], leading to significant deviation in relation to the deterministic approach.

Figure 4 shows the graphs of the covariance and variance functions of the WHSP. From these graphs and Eq. (31) it can be seen that the functions increase as the sliding time increases. From a tribological point of view, this is a predictable result, as shown in Ref. [22].

As the covariance is a continuous, monotonically increasing function for a closed and restricted region $[0,T]\times[0,T]\subset\mathbb{R}^2$, it reaches a maximum at $(s,t)\in[0,T]\times[0,T]$.

Problem 2: Wear and roughness coefficients modeled by a stochastic process. In this problem the uncertainties in the wear and roughness coefficients are modeled by stationary stochastic processes. Formal solutions for three cases of covariance functions are presented.

(a) $C_k(s,t)=10^{-14}\cdot(1-|s-t|/T)$ and $C_\beta(s,t)=10^{-10}\cdot(1+|s-t|/T)\cdot e^{-|s-t|/T}$, $\forall(s,t)\in[0,T]\times[0,T]$

(b) $C_k(s,t)=10^{-14}\cdot(1-|s-t|/T)$ and $C_\beta(s,t)=10^{-10}\cdot e^{-|s-t/T|^2}$, $\forall(s,t)\in[0,T]\times[0,T]$

(c) $C_k(s,t)=10^{-14}\cdot(1+|s-t|/T)\cdot e^{-|s-t/T|}$ and $C_\beta(s,t)=10^{-10}\cdot e^{-|s-t/T|^2}$, $\forall(s,t)\in[0,T]\times[0,T]$

It should be mentioned that an explicit solution cannot be determined in some cases because the integral in Eq. (34) cannot be solved analytically. In these cases, Eq. (34) is integrated numerically, and the solutions in terms of covariance are shown graphically.

Figure 5 shows the graphs of the products of the covariance functions for cases (a), (b), and (c). To compare them with the graphs shown in Fig. 2, the functions generated by the product of the covariance functions will be normalized. To that end, $\tau=s-t/T$ is taken into account and $\tilde{C}(\tau)=(\sigma_k\cdot\sigma_\beta)^{-2}\cdot(C_k\cdot C_\beta)(\tau)$ is defined in $[-1/\sqrt{2},1/\sqrt{2}]$. When the graphs of functions $C(\cdot)$ and

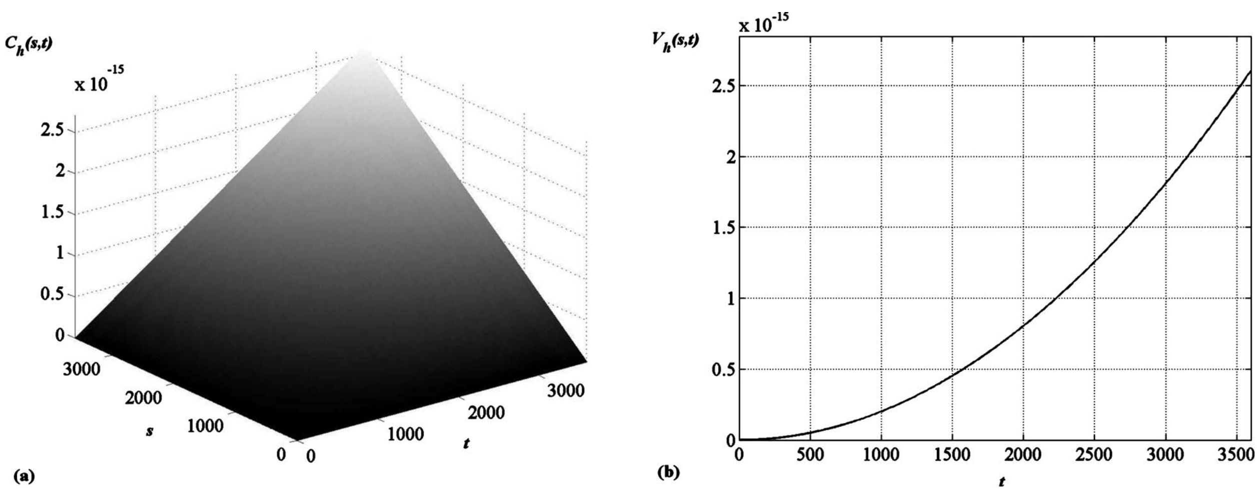

Fig. 4 (a) Covariance function (Eq. (28)); (b) Variance function

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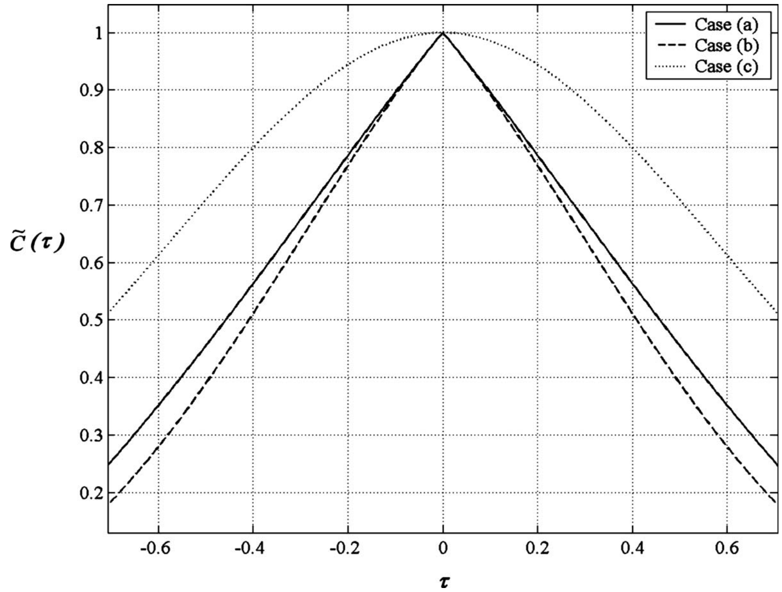

Fig. 5 Graphs of functions $\tilde{C}(\cdot)$ for cases (a), (b), and (c)

$\tilde{C}(\cdot)$ in Figs. 2 and 5, respectively, are compared, it can be observed that the function $\tilde{C}(\cdot)$ generally shows a smaller region of correlation than function $C(\cdot)$.

Figure 6 shows the graphs of the covariance functions of the WHSP for cases (a), (b), and (c). As expected, the graphs show a common trend, namely, an increase in the dispersion of the WHSP as the sliding time increases. It can also be observed that the covariance function had the highest values in case (a), a finding that is related to the volume $(\vartheta_{\tilde{C}})$ of the covariance function in each case. In addition, defining the region of correlation as the set $\Sigma_C = \{(s,t,z) \in \mathbb{R}^3 | (s,t) \in \text{supp}(C) \wedge 0 \le z \le C_h(s,t)\}$, one can conclude that

$$\Sigma_{C_{(b)}} \subseteq \Sigma_{C_{(a)}} \subseteq \Sigma_{C_{(c)}} \overset{\text{Eq.(34)}}{\Rightarrow} \vartheta_{C_{(b)}} \le \vartheta_{C_{(a)}} \le \vartheta_{C_{(c)}}$$

Figure 7 shows the graphs of the variance functions of the WHSP for cases (a), (b), and (c). The variance was greatest for case (c), while the dispersion in terms of the variance of the WHSP was smallest for case (b).

Table 1 gives the maximum values for the covariance of the WHSP in problems 1 and 2. The results show that the smallest dispersion values were obtained when the uncertainty was modeled using a stochastic process, a trend that was observed by Ávila da Silva and Pintaude [11], who considered only the uncertainty in the wear coefficient.

The results for the expected value and covariance given in this section illustrate a number of important issues related to the modeling of parameters used in the Archard model. A larger dispersion was observed when a random variable was used in the uncertainty modeling rather than a stochastic process. Although modeling using a stochastic process required more complex mathematics than modeling with a random variable, it produced a smaller propagation of the uncertainty in the WHSP.

8 Conclusions

This paper has described a simple yet powerful application of mean square calculus to worn height based on the Archard model. The formulations described were based on the probabilistic approach to the wear coefficient and the initial value problem described in the paper by Ávila da Silva and Pintaude [11]. The modeling of the uncertainty in the wear and roughness coefficients was studied for two different cases: One in which random variables were used, and the other, stochastic processes. Using suitable hypotheses about the uncertainty models for these coefficients and a classic result of mean square calculus, the Cauchy problems for the expected value and covariance of the worn height stochastic process (WHSP) were solved for the expected value and autocorrelation of the WHSP. For these problems, formal solutions for the expected value and covariance of the WHSP were described. In the case where the modeling was based on a random variable, explicit solutions for these functions in terms of the expected values and standard deviations for the wear and roughness coefficients were given. In the case where a stochastic process was used, only one formal solution for the covariance of

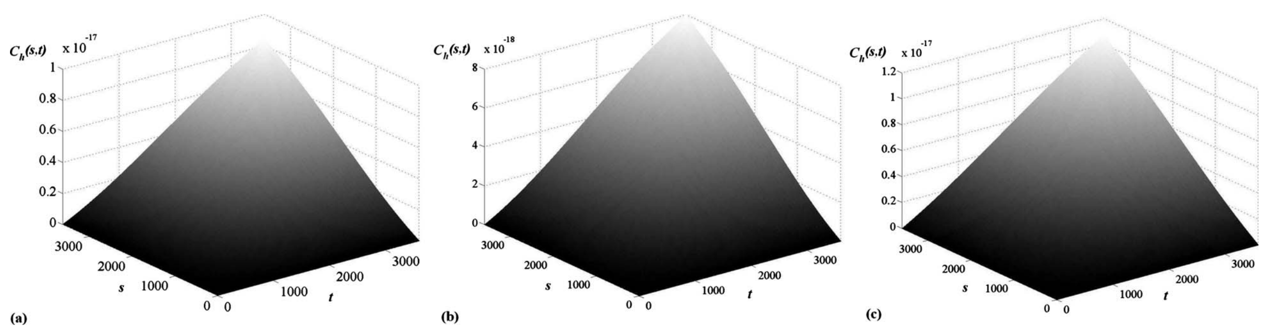

Fig. 6 Graphs of covariance functions of the WHSP for cases (a), (b), and (c)

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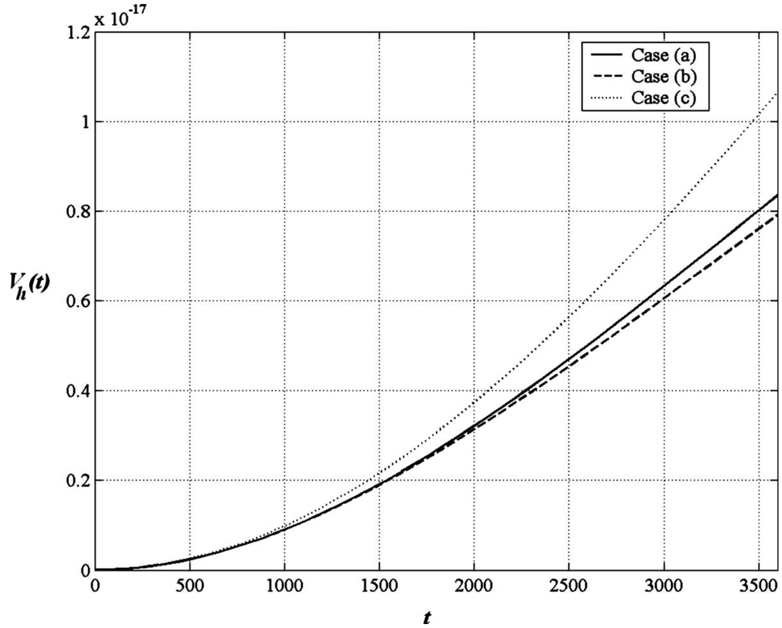

Fig. 7 Graphs of variance functions of the WHSP for cases (a), (b), and (c)

Table 1 Maximum value of covariance of the worn height stochastic process for problems 1 and 2

Covariance (m^2)	Problem 1	Problem 2		
		Case (a)	Case (b)	Case (c)
$C_h(3600,3600)$	2.60496×10^{-15}	8.35387479×10^{-18}	7.91247569×10^{-18}	1.06428137×10^{-17}

the worn height was presented. Numerical solutions for the WHSP were determined for the three models of covariance for the wear and roughness coefficients, and the covariance of the WHSP gave an indication of the uncertainty propagation present in these coefficients. A larger dispersion for the covariance of the worn height was observed for the uncertainty modeling based on a random variable.

Nomenclature

A_n = apparent contact area
A_y = plastic contact area
C_h = covariance function of the worn height
C_k = covariance function of the wear coefficient
C_β = covariance function of the roughness coefficient
$\tilde{C}$ = product of the covariance functions
h_0 = normalized initial height
h = reduction in height corresponding to the material removed
k = wear coefficient
P = probability measurement
R_h = autocorrelation of the stochastic process
$R_{dh/dt}$ = autocorrelation function of the random variables $dh/dt(s,\omega)$ and $dh/dt(t,\omega)$
T = total time interval
s or t = sliding test time
V_h = variance function of worn height
v = sliding velocity
β = roughness coefficient
F_ξ = probability density function (PDF)
μ_β = expected value of the roughness coefficient
μ_k = expected value of the wear coefficient
μ_h = expected value of the worn height
$\mu_{k\beta v}$ = expected value of random variable $(k\cdot\beta\cdot v)$
ϑ = volume of the covariance function
σ_β = standard deviation of the random variable β
σ_k = standard deviation of the random variable k
ζ,ξ,v = random variables
ω = element of the sample space Ω

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